Theorem: Let $A \in M_n(\mathbb{C})$.

Then $\exists$ unitary matrix $W \in M_n(\mathbb{C})$ and an upper-triangular matrix $T \in M_n(\mathbb{C})$ with

$$\boxed{A = WTW^*}$$

proof: By induction.

$n=1$, trivial.

We assumed true for $k=n-1$ and then showed that if x is an eigenvector corresponding to eigenvalue λ for A^*, then

$$A(\{x\}^\perp) \subseteq \{x\}^\perp.$$

Let

$$V : \{x\}^{\perp} \to \mathbb{C}^{n-1}$$

as follows: choose $\{y_1, y_2, \cdots, y_{n-1}\}$ an orthonormal basis for $\{x\}^{\perp}$. Define

$$\boxed{V\left(\sum_{i=1}^{n-1} \alpha_i y_i\right) = \sum_{i=1}^{n-1} \alpha_i e_i.}$$

Let $A_1 \in M_{n-1}(\mathbb{C})$,

$$A_1 = V A V^{-1}.$$

Since $A(\{x\}^{\perp}) \subseteq \{x\}^{\perp}$,

this is well-defined.

By induction, $\exists$ orthonormal basis $\{z_1, z_2, \dots, z_{n-1}\}$ of $\mathbb{C}^{n-1}$ under which A_1 is upper-triangular.

Define, $\forall\ 1 \leq i \leq n-1$,

$$x_i = V^{-1} z_i \quad \text{and}$$

$$x_n = x.$$ (assume x normalized)

Let $W : \mathbb{C}^n \to \mathbb{C}^n$,

$$W(e_i) = x_i,$$

$1 \leq i \leq n.$

Then W is a unitary

Claim: $W^* A W$ is upper-triangular.

$$\langle W^* A W e_i, e_n \rangle \quad (1 \leq i \leq n-1)$$

$$= \langle A W e_i, W e_n \rangle$$

$$= \langle A x_i, x_n \rangle$$

$$= \langle A x_i, x \rangle.$$

But $x_i \in \{x\}^\perp \; \forall \; 1 \leq i \leq n-1$,

and $A(\{x\}^\perp) \subseteq \{x\}^\perp$, so

$A x_i \in \{x\}^\perp \; \forall \; 1 \leq i \leq n-1$

$$\Rightarrow \langle A x_i, x \rangle = 0$$

$$\forall \; 1 \leq i \leq n-1$$

$$\Rightarrow \boxed{\begin{gathered} \langle W^* A W e_i, e_n \rangle = 0 \\ \forall \; 1 \leq i \leq n-1. \end{gathered}}$$

This shows

$$(w^* A w)_{n,i} = 0 \quad \forall\, 1 \leq i \leq n-1.$$

Now we want to show that

if $1 \leq j < i \leq n-1$, then

$$(w^* A w)_{i,j} = 0$$

$$(w^* A w)_{i,j}$$

$$= \langle w^* A w e_j, e_i \rangle$$

$$= \langle A w e_j, w e_i \rangle$$

$$= \langle A x_j, x_i \rangle$$

$$= \langle A v^{-1} z_j, v^{-1} z_i \rangle_{\mathbb{C}^n}$$

$$= \langle v A v^{-1} z_j, z_i \rangle_{\mathbb{C}^{n-1}}$$

$= \langle A_1 z_j, z_i \rangle$

$= 0$ since A_1

is upper-triangular

with respect to

$\{z_1, z_2, \ldots, z_{n-1}\}$.

This would show A is

upper-triangular in

$\{x_1, x_2, \ldots, x_n\}$.

But why is

$$\langle AV^{-1} z_j, V^{-1} z_i \rangle_{\mathbb{C}^n}$$

$$= \langle VAV^{-1} z_j, z_i \rangle_{\mathbb{C}^{n-1}} ?$$

Well, if $s, t \in \mathbb{C}^{n-1}$,

then if $s = \sum_{i=1}^{n-1} \alpha_i e_i$

and $t = \sum_{i=1}^{n-1} \beta_i e_i$, then

$$\langle VAV^{-1}s, t\rangle_{\mathbb{C}^{n-1}}$$

$$= \left\langle VA\sum_{i=1}^{n-1}\alpha_i y_i, \sum_{i=1}^{n}\beta_i e_i\right\rangle_{\mathbb{C}^{n-1}}$$

$$= \left\langle V\sum_{i=1}^{n-1}\alpha_i Ay_i, \sum_{i=1}^{n}\beta_i e_i\right\rangle_{\mathbb{C}^{n-1}}$$

$$= \left\langle V\sum_{i=1}^{n-1}\alpha_i \sum_{j=1}^{n-1}\gamma_{i,j} y_j, \sum_{i=1}^{n}\beta_i e_i\right\rangle_{\mathbb{C}^{n-1}}$$

$$= \boxed{\left\langle \sum_{i=1}^{n-1}\alpha_i \sum_{j=1}^{n-1}\gamma_{i,j} e_j, \sum_{i=1}^{n}\beta_i e_i\right\rangle}$$

But

$$\langle A v^{-1} s, v^{-1} t \rangle_{\mathbb{C}^n}$$

$$= \left\langle A\left(\sum_{i=1}^{n-1} \alpha_i y_i\right), \sum_{i=1}^{n-1} \beta_i y_i \right\rangle_{\mathbb{C}^n}$$

$$= \left\langle \sum_{i=1}^{n-1} \alpha_i A y_i, \sum_{i=1}^{n-1} \beta_i y_i \right\rangle_{\mathbb{C}^n}$$

$$= \boxed{\left\langle \sum_{i=1}^{n-1} \alpha_i \sum_{j=1}^{n-1} \gamma_{i,j} y_j, \sum_{i=1}^{n-1} \beta_i y_i \right\rangle}$$

Since

$\{y_i\}_{i=1}^{n-1}$ is orthonormal

in $\{x\}^{\perp}$, we obtain that

$$\langle A v^{-1} s, v^{-1} t \rangle_{\mathbb{C}^n}$$

$$= \langle v A v^{-1} s, t \rangle_{\mathbb{C}^{n-1}},$$

which completes the proof. $\square$

Definition: (absolute value)

Let $A \in M_n(\mathbb{C})$.

Then A^*A is

self-adjoint and positive

semi-definite. By results

in class and from HW #5,

A^*A is unitarily diagonalizable,

$$\boxed{A^*A = WDW^*}$$

where W is unitary

and D is diagonal

with $D = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}$

and $\boxed{d_i \geq 0} \ \forall \ 1 \leq i \leq n$

Define the absolute value

of A to be

$$\boxed{|A| = W \begin{pmatrix} \sqrt{d_1} & & & 0 \\ & \sqrt{d_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{d_n} \end{pmatrix} W^*}$$

Note

$$|A|^2 = W\begin{pmatrix} \sqrt{d_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{d_n} \end{pmatrix}\underbrace{\cancel{W^* W}}_{I_n}\begin{pmatrix} \sqrt{d_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{d_n} \end{pmatrix}W^*$$

$$= W\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}W^*$$

$$= A^* A .$$

Theorem: (polar decomposition)

Let $A \in M_n(\mathbb{C})$. Then

$\exists$ unitary $U \in M_n(\mathbb{C})$

with

$$\boxed{A = U|A|}$$